

Fubini theorem in noncommutative geometry[☆]

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Abstract

We discuss the Fubini formula in Alain Connes' noncommutative geometry. We present a sufficient condition on spectral triples for which a Fubini formula holds true. The condition is natural and related to heat semigroup asymptotics. We provide examples of spectral triples for which the Fubini formula fails.

1. Introduction

Fix throughout a separable infinite dimensional Hilbert space H . We let $\mathcal{L}(H)$ denote the algebra of all bounded operators on H . For a compact operator T on H , let $\lambda(k, T)$ and $\mu(k, T)$ denote its k -th eigenvalue¹ and k -th singular value (these are the eigenvalues of $|T|$).

We let $\mathcal{L}_{1,\infty}$ denote the principal ideal in $\mathcal{L}(H)$ generated by the operator $\text{diag}(\{\frac{1}{k+1}\}_{k \geq 0})$. Equivalently,

$$\mathcal{L}_{1,\infty} = \{T \in \mathcal{L}(H) : \mu(k, T) = O(\frac{1}{k+1})\}.$$

Note that our notation differs from the one used in [4].

The following result is proposed on p. 563 in [4].

Proposition 1.1. *Let $(1 + D_1^2)^{-p_1/2}, (1 + D_2^2)^{-p_2/2} \in \mathcal{L}_{1,\infty}$ and let $T_1, T_2 \in \mathcal{L}(H)$. If one of the elements $T_1(1 + D_1^2)^{-p_1/2}, T_2(1 + D_2^2)^{-p_2/2}$ is convergent,*

[☆]Research supported by the ARC.

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¹The eigenvalues are counted with algebraic multiplicities and arranged so that their absolute values are non-increasing.

then

$$\begin{aligned} & \Gamma(1 + \frac{p_1 + p_2}{2}) \text{Tr}_\omega((T_1 \otimes T_2)(1 + D_1^2 \otimes 1 + 1 \otimes D_2^2)^{-(p_1 + p_2)/2}) = \\ & = \Gamma(1 + \frac{p_1}{2}) \text{Tr}_\omega(T_1(1 + D_1^2)^{-p_1/2}) \cdot \Gamma(1 + \frac{p_2}{2}) \text{Tr}_\omega(T_2(1 + D_2^2)^{-p_2/2}) \quad (1.1) \end{aligned}$$

holds for some (Dixmier) trace Tr_ω on $\mathcal{L}_{1,\infty}$.

The wording on p. 563 in [4] is that “one of the two terms is convergent” is open for interpretation. One possible interpretation is that the operator $T_1(1 + D_1^2)^{-p_1/2}$ (or $T_2(1 + D_2^2)^{-p_2/2}$) is Tauberian. Recall that an operator $A \in \mathcal{L}_{1,\infty}$ is called Tauberian if there exists a limit

$$\lim_{n \rightarrow \infty} \frac{1}{\log(n+2)} \sum_{k=0}^n \lambda(k, A) = c$$

or, in a form convenient for comparison further below,

$$\sum_{k=0}^n \lambda(k, A) = c \cdot \log(n+1) + o(\log(n+1)).$$

We have therefore rephrased the proposition as one of the two operators is Tauberian.

The functional

$$T \mapsto \Gamma(1 + \frac{p}{2}) \text{Tr}_\omega(T(1 + D^2)^{-p/2})$$

is considered the p -dimensional integral in Connes’ noncommutative geometry [4]. That $T(1 + D^2)^{-p/2}$ is Tauberian implies that the functional is independent of which Dixmier trace Tr_ω is used to define the functional, a property called measurability. The result proposed on p. 563 in [4] is a Fubini formulation for noncommutative geometry, emulating the classical Fubini theorem where the integral on the product space is calculated from the product of the integrals provided one of the integral exists.

In recent personal communication, Professor Connes has kindly explained to the authors that the convergence he had in mind is “the convergence in the theta function formula (which in [4] is 4 lines above 2.Example a)). The assumed theta convergence is clearly stronger than the convergence of its Cesaro means and all the counter examples of the paper are about this nuance. As shown in Lemma 1.9 and Theorem 1.10, the Fubini formula indeed holds under theta convergence, by paying attention to the choice of the limiting processes, this is due to Professor Connes and we are grateful for his permission to include his proof into the paper.”

Our aim is to study the Fubini formula in detail. We show that the proposal in Proposition 1.1 does not hold under the condition that one of the terms is Tauberian. It does not hold either with an amended condition that both terms

are Tauberian. It does not hold if we ask if one or both of the terms satisfy the stronger condition that

$$\sum_{k=0}^n \lambda(k, T(1 + D^2)^{-p/2}) = c \cdot \log(n + 1) + O(1).$$

However, we show that there are natural conditions on the terms such that the Fubini formula as stated does hold. As explained above, one of them (see Condition 1.8 below) is also due to Professor Connes.

To state our results we need some definitions. The following terminology was recently introduced in [2].

Definition 1.2. *An operator $A \in \mathcal{L}_{1,\infty}$ is universally measurable if $\varphi(A)$ does not depend on the normalised² trace φ on $\mathcal{L}_{1,\infty}$. Equivalently (see Theorem 2.3)*

$$\sum_{k=0}^n \lambda(k, A) = c \cdot \log(n + 1) + O(1).$$

Clearly being universally measurable is stronger than being Tauberian. Proposition 1.1 is false if we show that the same proposition is false for universally measurable operators.

Definition 1.3. *We say that a (p, ∞) -summable spectral triple³ (\mathcal{A}, H, D) admits a noncommutative integral if, for every $T \in \mathcal{A}$, the operator $T(1 + D^2)^{-p/2}$ is universally measurable. In this case, we set*

$$\oint(T) = \varphi(T(1 + D^2)^{-p/2}), \quad T \in \mathcal{A}$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

The following condition is an analogue of the heat semigroup asymptotics found in Lemma 1.9.2 in [10]. It is satisfied by all commutative spectral triples of Riemannian manifolds (see the proof of Proposition 3.23 and Theorem 3.24 in [17]). Noncommutative tori also satisfies this condition (see the proof of Corollary 1.6).

Condition 1.4. *(\mathcal{A}, H, D) is a (p, ∞) -summable spectral triple such that, for every $T \in \mathcal{A}$, there exists $\varepsilon > 0$ such that*

$$\mathrm{Tr}(Te^{-(tD)^2}) = \frac{c(T)}{t^p} + O\left(\frac{1}{t^{p-\varepsilon}}\right), \quad t \rightarrow 0. \quad (1.2)$$

Our main Fubini theorem can be stated as follows.

²A trace on $\mathcal{L}_{1,\infty}$ is a unitarily invariant linear functional on $\mathcal{L}_{1,\infty}$. It is normalised if $\varphi(\mathrm{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \dots\})) = 1$.

³We refer to [4] for the definition of a spectral triple.

Theorem 1.5. *Suppose the spectral triples $(\mathcal{A}_1, H_1, D_1)$ and $(\mathcal{A}_2, H_2, D_2)$ satisfy the Condition 1.4. Then*

- (a) $(\mathcal{A}_1, H_1, D_1)$ and $(\mathcal{A}_2, H_2, D_2)$ admit a noncommutative integral.
- (b) $(\mathcal{A}_1 \otimes \mathcal{A}_2, H_1 \otimes H_2, (D_1^2 \otimes 1 + 1 \otimes D_2^2)^{1/2})$ satisfies the Condition 1.4 and admits a noncommutative integral.
- (c) For every $T_1 \in \mathcal{A}_1$, $T_2 \in \mathcal{A}_2$, and for every normalised trace φ on $\mathcal{L}_{1,\infty}$, we have

$$\begin{aligned} & \Gamma\left(1 + \frac{p_1 + p_2}{2}\right) \varphi((T_1 \otimes T_2)(1 + D_1^2 \otimes 1 + 1 \otimes D_2^2)^{-\frac{p_1 + p_2}{2}}) = \\ & = \Gamma\left(1 + \frac{p_1}{2}\right) \varphi(T_1(1 + D_1^2)^{-\frac{p_1}{2}}) \cdot \Gamma\left(1 + \frac{p_2}{2}\right) \varphi(T_2(1 + D_2^2)^{-\frac{p_2}{2}}). \end{aligned}$$

In particular, a Fubini formula holds for noncommutative tori, for sphere \mathbb{S}^2 and for the quantum group $SU_q(2)$. The proofs for noncommutative tori extend the idea used in the proof of the main result of [15].

Corollary 1.6. *Let $(\mathcal{A}_1, H_1, D_1)$ and $(\mathcal{A}_2, H_2, D_2)$ be noncommutative tori. For every $T_1 \in \mathcal{A}_1$, $T_2 \in \mathcal{A}_2$, and for every normalised trace φ on $\mathcal{L}_{1,\infty}$, we have*

$$\begin{aligned} & \Gamma\left(1 + \frac{p_1 + p_2}{2}\right) \varphi((T_1 \otimes T_2)(1 + D_1^2 \otimes 1 + 1 \otimes D_2^2)^{-\frac{p_1 + p_2}{2}}) = \\ & = \Gamma\left(1 + \frac{p_1}{2}\right) \varphi(T_1(1 + D_1^2)^{-\frac{p_1}{2}}) \cdot \Gamma\left(1 + \frac{p_2}{2}\right) \varphi(T_2(1 + D_2^2)^{-\frac{p_2}{2}}). \end{aligned}$$

Corollary 1.7. *Let $(\mathcal{A}_1, H_1, D_1)$ and $(\mathcal{A}_2, H_2, D_2)$ be spectral triples which correspond either to sphere \mathbb{S}^2 or to the quantum group $SU_q(2)$. For every $T_1 \in \mathcal{A}_1$, $T_2 \in \mathcal{A}_2$, and for every normalised trace φ on $\mathcal{L}_{1,\infty}$, we have*

$$\begin{aligned} & \Gamma\left(1 + \frac{p_1 + p_2}{2}\right) \varphi((T_1 \otimes T_2)(1 + D_1^2 \otimes 1 + 1 \otimes D_2^2)^{-\frac{p_1 + p_2}{2}}) = \\ & = \Gamma\left(1 + \frac{p_1}{2}\right) \varphi(T_1(1 + D_1^2)^{-\frac{p_1}{2}}) \cdot \Gamma\left(1 + \frac{p_2}{2}\right) \varphi(T_2(1 + D_2^2)^{-\frac{p_2}{2}}). \end{aligned}$$

Condition 1.8, Theorems 1.9 and 1.10 below were suggested by Professor Connes. We are grateful for this valuable addition to the paper.

Condition 1.8. (\mathcal{A}, H, D) is a (p, ∞) -summable spectral triple such that, for every $T \in \mathcal{A}$, we have

$$t^p \text{Tr}(T e^{-(tD)^2}) \rightarrow c(T), \quad t \rightarrow 0. \quad (1.3)$$

In what follows, we use a notation $\omega^u = \omega \circ P_u$, $u > 0$. We refer the reader to Section 2 for the definition of Dixmier traces.

Theorem 1.9. *Let $\omega = \omega \circ M$ be a state on $L_\infty(0, \infty)$. Suppose that the spectral triple $(\mathcal{A}_1, H_1, D_1)$ (or $(\mathcal{A}_2, H_2, D_2)$) satisfies Condition 1.8. For every $T_1 \in \mathcal{A}_1$, $T_2 \in \mathcal{A}_2$, we have*

$$\begin{aligned} & \Gamma\left(1 + \frac{p_1 + p_2}{2}\right) \text{Tr}_{\omega^{p_1 + p_2}}((T_1 \otimes T_2)(1 + D_1^2 \otimes 1 + 1 \otimes D_2^2)^{-\frac{p_1 + p_2}{2}}) = \\ & = \Gamma\left(1 + \frac{p_1}{2}\right) \text{Tr}_{\omega^{p_1}}(T_1(1 + D_1^2)^{-\frac{p_1}{2}}) \cdot \Gamma\left(1 + \frac{p_2}{2}\right) \text{Tr}_{\omega^{p_2}}(T_2(1 + D_2^2)^{-\frac{p_2}{2}}). \end{aligned}$$

Theorem 1.9 allows us to state another version of Fubini theorem as follows.

Theorem 1.10. *Suppose that the spectral triple $(\mathcal{A}_1, H_1, D_1)$ (or $(\mathcal{A}_2, H_2, D_2)$) satisfies the Condition 1.8. For every $T_1 \in \mathcal{A}_1$, $T_2 \in \mathcal{A}_2$, and for every Dixmier trace $\text{Tr}_\omega \in \mathfrak{M}^4$ on $\mathcal{L}_{1,\infty}$, we have*

$$\begin{aligned} & \Gamma\left(1 + \frac{p_1 + p_2}{2}\right) \text{Tr}_\omega((T_1 \otimes T_2)(1 + D_1^2 \otimes 1 + 1 \otimes D_2^2)^{-\frac{p_1 + p_2}{2}}) = \\ & = \Gamma\left(1 + \frac{p_1}{2}\right) \text{Tr}_\omega(T_1(1 + D_1^2)^{-\frac{p_1}{2}}) \cdot \Gamma\left(1 + \frac{p_2}{2}\right) \text{Tr}_\omega(T_2(1 + D_2^2)^{-\frac{p_2}{2}}). \end{aligned}$$

It is important to note the difference between Conditions 1.4 and 1.8 and the difference between the assertions of Theorems 1.5 and 1.10. Indeed, Theorem 1.5 holds for arbitrary traces on $\mathcal{L}_{1,\infty}$, while Theorem 1.10 holds for a certain subclass \mathfrak{M} in the class of Dixmier traces. Theorem 1.10 does not hold for some Dixmier traces outside of the subclass \mathfrak{M} .

Condition 1.4 is stronger than universal measurability. Our second result complements Theorem 1.5 by stating that universal measurability is not sufficient for a Fubini theorem. In fact, the counterexample involves the nicest possible situation where the noncommutative integral is a normal functional on the algebra \mathcal{A} .

Definition 1.11. *Suppose that (\mathcal{A}, H, D) admits a noncommutative integral. We say that the noncommutative integral is normal if the mapping*

$$T \rightarrow \oint(T), \quad T \in \mathcal{A},$$

is continuous in the weak operator topology.

Theorem 1.12. *There exists a $(1, \infty)$ -summable spectral triple (\mathcal{A}, H, D) such that*

- (a) D has simple spectrum \mathbb{Z}_+ .
- (b) \mathcal{A} is generated by a unitary operator U and is finite dimensional.
- (c) (\mathcal{A}, H, D) admits a normal noncommutative integral.
- (d) $(\mathcal{A} \otimes \mathcal{A}, H \otimes H, (D^2 \otimes 1 + 1 \otimes D^2)^{1/2})$ admits a normal noncommutative integral.
- (e) For every normalised trace φ on $\mathcal{L}_{1,\infty}$, we have

$$\varphi((U \otimes U^{-1})(1 + D^2 \otimes 1 + 1 \otimes D^2)^{-1}) \neq 0, \quad \varphi(U(1 + D^2)^{-\frac{1}{2}}) = \varphi(U^{-1}(1 + D^2)^{-\frac{1}{2}}) = 0.$$

Corollary 1.13. *In the setting of Theorem 1.12, there exists a positive element $T \in \mathcal{A}$ such that*

$$\varphi((T \otimes T)(1 + D^2 \otimes 1 + 1 \otimes D^2)^{-1}) \neq \frac{\pi}{4}(\varphi(T(1 + D^2)^{-\frac{1}{2}}))^2$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

⁴Here, \mathfrak{M} is a subclass of Dixmier traces specified in the next section.

Our second counterexample shows that volume in noncommutative geometry is not necessarily well behaved under the product operation on spectral triples, even with the strong condition of universal measurability.

Theorem 1.14. *There exists an operator D such that*

- (a) $(1 + D^2)^{-1/2} \in \mathcal{L}_{1,\infty}$ *is universally measurable.*
- (b) $(1 + D^2 \otimes 1 + D^2 \otimes 1)^{-1} \in \mathcal{L}_{1,\infty}$ *is universally measurable.*
- (c) *For every normalised trace φ on $\mathcal{L}_{1,\infty}$, we have*

$$\varphi((1 + D^2 \otimes 1 + 1 \otimes D^2)^{-1}) > \frac{\pi}{4}(\varphi((1 + D^2)^{-\frac{1}{2}}))^2.$$

Our final counterexample (proved in Appendix Appendix B) shows that it does not suffice to impose Condition 1.4 only on one spectral triple. It also shows that the assertion of Theorem 1.10 fails for some Dixmier trace (outside of the class \mathfrak{M}).

Theorem 1.15. *There exist spectral triples (\mathcal{A}_1, l_2, D) and (\mathbb{C}, l_2, D) such that*

- (a) *D has simple spectrum \mathbb{Z}_+ . In particular, the spectral triple (\mathbb{C}, l_2, D) satisfies the Condition 1.4.*
- (b) *There exists an operator $T_1 \in \mathcal{A}_1$ and a (Dixmier) trace φ on $\mathcal{L}_{1,\infty}$ such that*

$$\varphi((T_1 \otimes 1)(1 + D^2 \otimes 1 + 1 \otimes D^2)^{-1}) \neq \frac{\pi}{4}\varphi(T_1(1 + D^2)^{-1/2}).$$

We are grateful to Professor Connes for his kind explanation of noncommutative Fubini formula. We are also grateful to A. Carey, Y. Kuznetsova, S. Lord and A. Rennie for useful comments on the earlier versions of this manuscript. The authors would also like to mention that the idea of the proof of Lemma 5.1 was conveyed to them by the late N. Kalton.

2. Preliminaries

The standard trace on $\mathcal{L}(H)$ is denoted by Tr . Fix an orthonormal basis in H (the particular choice of basis is inessential). We identify the algebra l_∞ of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. We set $l_{1,\infty} = \mathcal{L}_{1,\infty} \cap l_\infty$. For a given sequence $x \in l_\infty$, we denote the corresponding diagonal operator by $\text{diag}(x)$.

Definition 2.1. *A trace on $\mathcal{L}_{1,\infty}$ is a unitarily invariant linear functional $\varphi : \mathcal{L}_{1,\infty} \rightarrow \mathbb{C}$.*

Traces on $\mathcal{L}_{1,\infty}$ satisfying the condition

$$\varphi(TS) = \varphi(ST), \quad T \in \mathcal{L}_{1,\infty}, S \in \mathcal{L}(H).$$

The latter may be reinterpreted as the vanishing of the linear functional φ on the commutator subspace

$$[\mathcal{L}_{1,\infty}, \mathcal{L}(H)] = \text{span}\{ST - TS, \quad T \in \mathcal{L}_{1,\infty}, S \in \mathcal{L}(H)\}.$$

An example of a trace on $\mathcal{L}_{1,\infty}$ is a Dixmier trace that we now explain (we use the definition from [18], which, according to Theorem 17 in [18], produces exactly the same class of traces on $\mathcal{L}_{1,\infty}$ as the one in [4]). Namely, for every ultrafilter ω , the functional Tr_ω defined on the positive cone of $\mathcal{L}_{1,\infty}$ by the formula

$$\text{Tr}_\omega(A) = \lim_{n \rightarrow \omega} \frac{1}{\log(n+2)} \sum_{k=0}^n \mu(k, A), \quad 0 \leq A \in \mathcal{L}_{1,\infty}, \quad (2.1)$$

is additive and, therefore, extends to a positive unitarily invariant linear functional on $\mathcal{L}_{1,\infty}$ called a Dixmier trace.

In order to properly state Theorem 1.10, we need a smaller subclass \mathfrak{M} of Dixmier traces. Let ω be a state on the algebra $L_\infty(0, \infty)$ which satisfies the condition $\omega = \omega \circ M$ (see p.35 in [1]). Here, the linear operator $M : L_\infty(0, \infty) \rightarrow L_\infty(0, \infty)$ is given by the formula

$$(Mx)(t) = \frac{1}{\log(t)} \int_1^t x(s) \frac{ds}{s}, \quad t > 0.$$

The functional Tr_ω is defined on the positive cone of $\mathcal{L}_{1,\infty}$ by the formula

$$\text{Tr}_\omega(A) = \omega\left(t \rightarrow \frac{1}{\log(1+t)} \int_0^t \mu(s, A) ds\right), \quad 0 \leq A \in \mathcal{L}_{1,\infty}.$$

This functional is additive and, therefore, extends to a positive unitarily invariant linear functional on $\mathcal{L}_{1,\infty}$ (see e.g. [1]).

Let the group (\mathbb{R}_+, \cdot) act on $L_\infty(0, \infty)$ by the formula $u \rightarrow P_u$, $(P_u x)(t) = x(t^u)$, $u, t > 0$. Note that $M \circ P_u = P_u \circ M$ (see a similar formula (3) in [19]). In particular, $\text{Tr}_{\omega^u} = \text{Tr}_{\omega \circ P_u}$ is also a positive unitarily invariant linear functional on $\mathcal{L}_{1,\infty}$. We set

$$\mathfrak{M} = \{\text{Tr}_\omega : \omega = \omega \circ M, \quad \omega = \omega \circ P_u, \quad u > 0\}.$$

It is important to note that ω in this paragraph can never be an ultrafilter. However, Tr_ω is still a Dixmier trace according to the main result of [18].

The following assertion is Theorem 3 in [1].

Theorem 2.2. *Let $\omega = \omega \circ M$ be a state on the algebra $L_\infty(0, \infty)$. If the triple (\mathcal{A}, H, D) is (p, ∞) -summable, then*

$$\Gamma(1 + \frac{p}{2}) \text{Tr}_{\omega \circ P_p}(T(1 + D^2)^{-\frac{p}{2}}) = \omega\left(t \rightarrow t^{-p} \text{Tr}(T e^{-t^{-2} D^2})\right), \quad T \in \mathcal{A}.$$

The following theorem provides the convenient spectral description for universally measurable operators referred to earlier. It was originally proved in [7] for normal operators and, then in [12] and [8] for arbitrary operators (see also [13]). For accessible proof, we refer the interested reader to Theorem 10.1.3 in [16] and its proof in Chapter 5 in [16].

Theorem 2.3. *For $A \in \mathcal{L}_{1,\infty}$, the following conditions are equivalent.*

- (a) *We have $\varphi(A) = c$ for every normalised trace φ on $\mathcal{L}_{1,\infty}$.*
- (b) *We have*

$$\sum_{m=0}^n \lambda(m, A) = c \cdot \log(n+1) + O(1), \quad n \geq 0.$$

In particular, $A \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$ if and only if

$$\sum_{m=0}^n \lambda(m, A) = O(1), \quad n \geq 0.$$

Every universally measurable operator is Tauberian (that is, Dixmier-measurable [18]). For various sorts of measurability results in noncommutative geometry, we refer the interested reader to papers [19, 20] and to the book [16].

3. Proof of Theorems 1.5 and 1.10

Lemma 3.1. *Let $\Phi : (0, \infty) \rightarrow (0, 1)$ be such that*

- (i) *Φ is convex, decreasing and positive.*
- (ii) *$\Phi(0) = 1$ and*

$$\int_1^\infty \Phi(t) \frac{dt}{t} < \infty, \quad \int_0^1 \left(\frac{1}{t} - 1\right)(1 - \Phi(t)) dt < \infty.$$

For every $0 \leq V \in \mathcal{L}_{1,\infty}$, we have

$$\|\min\{V\Phi((nV)^{-1}), \frac{1}{n}\}\|_1 = O(1), \quad n \rightarrow \infty$$

$$\|(V - \frac{1}{n})_+(1 - \Phi((nV)^{-1}))\|_1 = O(1), \quad n \rightarrow \infty.$$

Proof. It is easy using (i) to check that the functions

$$x \rightarrow x\Phi(x^{-1}), \quad x \rightarrow (x-1)_+(1 - \Phi(x^{-1}))$$

increase on $(0, \infty)$. For simplicity of computations, let $\|V\|_{1,\infty} = 1$. Let $W \in \mathcal{L}_{1,\infty}$ be an operator commuting with V such that $0 \leq V \leq W$ and such that $\mu(k, W) = \frac{1}{k+1}$, $k \geq 0$. It follows that

$$\begin{aligned} \|\min\{V\Phi((nV)^{-1}), \frac{1}{n}\}\|_1 &\leq \|\min\{W\Phi((nW)^{-1}), \frac{1}{n}\}\|_1 \leq \\ &\leq \|\{\frac{1}{k+1}\Phi(\frac{k+1}{n})\}_{k=n}^\infty\|_1 + \|\{\frac{1}{n}\}_{k=0}^{n-1}\|_1 \leq \int_n^\infty \Phi(\frac{t}{n}) \frac{dt}{t} + 1 \stackrel{(ii)}{=} O(1) \end{aligned}$$

and, similarly,

$$\|(V - \frac{1}{n})_+(1 - \Phi((nV)^{-1}))\|_1 \leq \|(W - \frac{1}{n})_+(1 - \Phi((nW)^{-1}))\|_1 \leq$$

$$\leq \|\{(\frac{1}{k+1} - \frac{1}{n})(1 - \Phi(\frac{k+1}{n}))\}_{k=0}^{n-1}\|_1 \leq \int_0^n (\frac{1}{t} - \frac{1}{n})(1 - \Phi(\frac{t}{n}))dt \stackrel{(ii)}{=} O(1).$$

□

The following lemma extends Proposition 6 in [2].

Lemma 3.2. *Let $0 \leq V \in \mathcal{L}_{1,\infty}$ and let $A \in \mathcal{L}(H)$. Let Φ be as in Lemma 3.1. The following conditions are equivalent*

- (a) $\varphi(AV) = c$ for every normalised trace φ on $\mathcal{L}_{1,\infty}$.
- (b) We have

$$\text{Tr}(AV\Phi((nV)^{-1})) = c \log(n) + O(1), \quad n \rightarrow \infty.$$

Proof. It is clear that

$$\begin{aligned} AV\Phi((nV)^{-1}) - AV E_V[\frac{1}{n}, \infty) &= \\ &= AV\Phi((nV)^{-1})E_V[0, \frac{1}{n}) + AV(\Phi((nV)^{-1}) - 1)E_V[\frac{1}{n}, \infty). \end{aligned}$$

Therefore,

$$\begin{aligned} |\text{Tr}(AV\Phi((nV)^{-1})) - \text{Tr}(AV E_V[\frac{1}{n}, \infty))| &\leq \|AV\Phi((nV)^{-1}) - AV E_V[\frac{1}{n}, \infty)\|_1 \leq \\ &\leq \|A\|_\infty \left(\|V\Phi((nV)^{-1})E_V[0, \frac{1}{n})\|_1 + \|V(\Phi((nV)^{-1}) - 1)E_V[\frac{1}{n}, \infty)\|_1 \right) \leq \\ &\leq \|A\|_\infty \left(\|V\Phi((nV)^{-1})E_V[0, \frac{1}{n})\|_1 + \|(V - \frac{1}{n})(\Phi((nV)^{-1}) - 1)E_V[\frac{1}{n}, \infty)\|_1 + \right. \\ &\quad \left. + \frac{1}{n}\|(\Phi((nV)^{-1}) - 1)E_V[\frac{1}{n}, \infty)\|_1 \right) \leq \\ &\leq \|A\|_\infty \left(\|\min\{V\Phi((nV)^{-1}), \frac{1}{n}\}\|_1 + \|(V - \frac{1}{n})_+(\Phi((nV)^{-1}) - 1)\|_1 + \frac{1}{n}\|E_V[\frac{1}{n}, \infty)\|_1 \right). \end{aligned}$$

It follows from Lemma 3.1 that

$$\text{Tr}(AV\Phi((nV)^{-1})) - \text{Tr}(AV E_V[\frac{1}{n}, \infty)) = O(1), \quad n \rightarrow \infty.$$

It follows now from Lemma 8 in [2] that

$$\text{Tr}(AV\Phi((nV)^{-1})) - \sum_{k=0}^n \lambda(k, AV) = O(1).$$

The assertion follows now from Theorem 2.3. □

Lemma 3.3. *If a spectral triple (\mathcal{A}, H, D) satisfies the Condition 1.4, then it admits a noncommutative integral. More precisely, if (\mathcal{A}, H, D) is (p, ∞) -summable, then*

$$c(T) = \Gamma(1 + \frac{p}{2})\varphi(T(1 + D^2)^{-\frac{p}{2}}), \quad T \in \mathcal{A},$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$. Here, $c(T)$ is the number which appears in (1.2).

Proof. It follows from (1.2) that

$$\mathrm{Tr}(Te^{-t^2(1+D^2)}) = \frac{c(T)}{t^p} + O\left(\frac{1}{t^{p-\varepsilon}}\right), \quad t \rightarrow 0.$$

Substituting $t^{\frac{1}{p}}$ instead of t , we infer that

$$\mathrm{Tr}(Te^{-(t(1+D^2)^{\frac{p}{2}})^{\frac{2}{p}}}) = \frac{c(T)}{t} + O\left(\frac{1}{t^{1-\varepsilon}}\right), \quad t \rightarrow 0. \quad (3.1)$$

Set

$$\Phi(s) = \frac{1}{\Gamma(1 + \frac{p}{2})} \int_s^\infty e^{-t^{\frac{2}{p}}} dt, \quad s > 0.$$

We have

$$\int_s^1 e^{-(t(1+D^2)^{\frac{p}{2}})^{\frac{2}{p}}} dt = (1+D^2)^{-p/2} \left(\Phi(s(1+D^2)^{p/2}) - \Phi((1+D^2)^{p/2}) \right),$$

where the integral is understood in the Bochner sense in \mathcal{L}_1 . In particular, we have

$$\begin{aligned} & \int_s^1 \mathrm{Tr}(Te^{-(t(1+D^2)^{\frac{p}{2}})^{\frac{2}{p}}}) dt = \\ &= \mathrm{Tr}(T(1+D^2)^{-\frac{p}{2}} \Phi(s(1+D^2)^{\frac{p}{2}})) - \mathrm{Tr}(T(1+D^2)^{-\frac{p}{2}} \Phi((1+D^2)^{\frac{p}{2}})). \end{aligned}$$

Integrating both sides in (3.1) over $t \in [s, 1]$ and dividing by $\Gamma(1 + \frac{p}{2})$ we infer that

$$\mathrm{Tr}(T(1+D^2)^{-\frac{p}{2}} \Phi(s(1+D^2)^{\frac{p}{2}})) = \frac{c(T)}{\Gamma(1 + \frac{p}{2})} \log\left(\frac{1}{s}\right) + O(1), \quad s \rightarrow 0.$$

Observe, that Φ satisfies the conditions of Lemma 3.1. The assertion follows now from Lemma 3.2 (as applied to $V = (1+D^2)^{-\frac{p}{2}}$ and $c = \frac{c(T)}{\Gamma(1 + \frac{p}{2})}$). \square

Proof of Theorem 1.5. Recall an abstract equality (which holds for all bounded operators T_1, T_2)

$$(T_1 \otimes T_2)e^{-t^2(D_1^2 \otimes 1 + 1 \otimes D_2^2)} = T_1 e^{-t^2 D_1^2} \otimes T_2 e^{-t^2 D_2^2}.$$

Take now $T_1 \in \mathcal{A}_1$ and $T_2 \in \mathcal{A}_2$. By Condition 1.4, we have

$$\mathrm{Tr}(T_1 e^{-t^2 D_1^2}) = \frac{c(T_1)}{t^{p_1}} + O\left(\frac{1}{t^{p_1-\varepsilon}}\right), \quad \mathrm{Tr}(T_2 e^{-t^2 D_2^2}) = \frac{c(T_2)}{t^{p_2}} + O\left(\frac{1}{t^{p_2-\varepsilon}}\right), \quad t \rightarrow 0.$$

It follows that

$$\mathrm{Tr}((T_1 \otimes T_2)e^{-t^2(D_1^2 \otimes 1 + 1 \otimes D_2^2)}) = \frac{c(T_1)c(T_2)}{t^{p_1+p_2}} + O\left(\frac{1}{t^{p_1+p_2-\varepsilon}}\right)$$

as $t \rightarrow 0$. Thus, the spectral triple $(\mathcal{A}_1 \otimes \mathcal{A}_2, H_1 \otimes H_2, (D_1^2 \otimes 1 + 1 \otimes D_2^2)^{1/2})$ satisfies the Condition 1.4 and

$$c(T_1 \otimes T_2) = c(T_1)c(T_2).$$

The assertion follows now from Lemma 3.3. \square

Proof of Theorem 1.9. Recall an abstract equality (which holds for all bounded operators T_1, T_2)

$$(T_1 \otimes T_2)e^{-t^2(D_1^2 \otimes 1 + 1 \otimes D_2^2)} = T_1 e^{-t^2 D_1^2} \otimes T_2 e^{-t^2 D_2^2}.$$

Take now $T_1 \in \mathcal{A}_1$ and $T_2 \in \mathcal{A}_2$. By Condition 1.8, we have

$$\mathrm{Tr}(T_1 e^{-t^2 D_1^2}) = \frac{c(T_1)}{t^{p_1}} + o\left(\frac{1}{t^{p_1}}\right), \quad t \rightarrow 0.$$

Taking the trace and replacing t with t^{-1} , we obtain that

$$t^{-(p_1+p_2)} \mathrm{Tr}((T_1 \otimes T_2)e^{-t^{-2}(D_1^2 \otimes 1 + 1 \otimes D_2^2)}) = (c(T_1) + o(1)) \cdot t^{-p_2} \mathrm{Tr}(T_2 e^{-t^{-2} D_2^2}), \quad t \rightarrow \infty.$$

In particular, applying ω to the both sides of the equality, we arrive at

$$\omega\left(t \rightarrow t^{-(p_1+p_2)} \mathrm{Tr}((T_1 \otimes T_2)e^{-t^{-2}(D_1^2 \otimes 1 + 1 \otimes D_2^2)})\right) = c(T_1) \omega\left(t \rightarrow t^{-p_2} \mathrm{Tr}(T_2 e^{-t^{-2} D_2^2})\right).$$

It follows from Theorem 2.2 (applied to both sides of the equality) that

$$\begin{aligned} \Gamma\left(1 + \frac{p_1 + p_2}{2}\right) \mathrm{Tr}_{\omega^{p_1+p_2}}((T_1 \otimes T_2)(1 + D_1^2 \otimes 1 + 1 \otimes D_2^2)^{-\frac{p_1+p_2}{2}}) = \\ = c(T_1) \cdot \Gamma\left(1 + \frac{p_2}{2}\right) \mathrm{Tr}_{\omega^{p_2}}(T_2(1 + D_2^2)^{-\frac{p_2}{2}}). \end{aligned}$$

Again using Theorem 2.2 (applied to the spectral triple $(\mathcal{A}_1, H_1, D_1)$), we infer that

$$c(T_1) = \Gamma\left(1 + \frac{p_1}{2}\right) \mathrm{Tr}_{\omega^{p_1}}(T_1(1 + D_1^2)^{-\frac{p_1}{2}}).$$

This concludes the proof. \square

Proof of Theorem 1.10. If $\omega = \omega \circ P_u$, $u > 0$, then

$$\mathrm{Tr}_{\omega \circ P_{p_1}} = \mathrm{Tr}_{\omega \circ P_{p_2}} = \mathrm{Tr}_{\omega \circ P_{p_1+p_2}} = \mathrm{Tr}_{\omega}.$$

The assertion follows now from Lemma 1.9. \square

4. Physically relevant examples

We supply 3 examples which satisfy the Condition 1.4. The first example is a sphere — the simplest possible non-flat manifold. The second example is a noncommutative torus. The third and the most technically involved example is the quantum group $\mathrm{SU}_q(2)$.

The following elementary lemma is needed in all 3 examples. We incorporate the proof for convenience of the reader.

Lemma 4.1. *We have*

(a)

$$\sum_{l \in \mathbb{Z}} |l| e^{-l^2 t^2} = t^{-2} + O(t^{-1}), \quad t \rightarrow 0.$$

(b)

$$\sum_{l \in \mathbb{Z}} e^{-l^2 t^2} = \frac{\pi^{\frac{1}{2}}}{t} + O(1), \quad t \rightarrow 0.$$

(c)

$$\sum_{l \in \mathbb{Z}} l^2 e^{-l^2 t^2} = \frac{\pi^{\frac{1}{2}}}{2t^3} + O(t^{-2}), \quad t \rightarrow 0.$$

Proof. Though the second and third equalities can be derived from the Poisson summation formula, this method gives nothing good for the first equality. We provide an elementary proof of the first equality. The proofs of the second and third are similar.

The function $s \rightarrow se^{-s^2 t^2}$ admits its maximum at the point $s = \frac{1}{t\sqrt{2}}$. Thus, the function increases on the interval $(0, \frac{1}{t\sqrt{2}})$ and decreases on the interval $(\frac{1}{t\sqrt{2}}, \infty)$. It follows that

$$\sum_{l=1}^{\lfloor \frac{1}{t\sqrt{2}} \rfloor - 1} \int_{l-1}^l se^{-s^2 t^2} ds \leq \sum_{l=0}^{\lfloor \frac{1}{t\sqrt{2}} \rfloor - 1} le^{-l^2 t^2} \leq \sum_{l=0}^{\lfloor \frac{1}{t\sqrt{2}} \rfloor - 1} \int_l^{l+1} se^{-s^2 t^2} ds.$$

Thus,

$$\left| \sum_{l=0}^{\lfloor \frac{1}{t\sqrt{2}} \rfloor - 1} le^{-l^2 t^2} - \int_0^{\lfloor \frac{1}{t\sqrt{2}} \rfloor} se^{-s^2 t^2} ds \right| \leq \int_{\lfloor \frac{1}{t\sqrt{2}} \rfloor - 1}^{\lfloor \frac{1}{t\sqrt{2}} \rfloor} se^{-s^2 t^2} ds \leq \sup_{s>0} se^{-s^2 t^2} = \frac{e^{-\frac{1}{2}}}{t\sqrt{2}}.$$

Similarly,

$$\left| \sum_{l=\lceil \frac{1}{t\sqrt{2}} \rceil + 1}^{\infty} le^{-l^2 t^2} - \int_{\lceil \frac{1}{t\sqrt{2}} \rceil}^{\infty} se^{-s^2 t^2} ds \right| \leq \frac{e^{-\frac{1}{2}}}{t\sqrt{2}}.$$

It follows that

$$\begin{aligned} \sum_{l=0}^{\infty} le^{-l^2 t^2} &= \left(\sum_{l=0}^{\lfloor \frac{1}{t\sqrt{2}} \rfloor - 1} le^{-l^2 t^2} \right) + \left(\sum_{l=\lceil \frac{1}{t\sqrt{2}} \rceil + 1}^{\infty} le^{-l^2 t^2} \right) + \left(\sum_{l=\lfloor \frac{1}{t\sqrt{2}} \rfloor}^{\lceil \frac{1}{t\sqrt{2}} \rceil} le^{-l^2 t^2} \right) = \\ &= \left(\int_0^{\lfloor \frac{1}{t\sqrt{2}} \rfloor} se^{-s^2 t^2} ds + O(t^{-1}) \right) + \left(\int_{\lceil \frac{1}{t\sqrt{2}} \rceil}^{\infty} se^{-s^2 t^2} ds + O(t^{-1}) \right) + O(t^{-1}) = \\ &= \int_0^{\infty} se^{-s^2 t^2} ds + O(t^{-1}) = t^{-2} \int_0^{\infty} se^{-s^2} ds + O(t^{-1}) = \frac{1}{2t^2} + O(t^{-1}). \end{aligned}$$

It follows that

$$\sum_{l=-\infty}^0 |l| e^{-l^2 t^2} = \sum_{l=0}^{\infty} l e^{-l^2 t^2} = \frac{1}{2t^2} + O(t^{-1}).$$

Adding the last 2 formulae, we conclude the proof. \square

4.1. Example: sphere \mathbb{S}^2

We briefly recall the construction of a spectral triple on sphere \mathbb{S}^2 . Interested reader is referred to [11] for details.

Let $s = (s_1, s_2, s_3)$ be the point on the sphere \mathbb{S}^2 expressed in Cartesian coordinates. Define stereographic coordinates by the formula $(x, y) = (\frac{s_1}{1-s_3}, \frac{s_2}{1-s_3})$. Denote $z = x + iy$ and $q(x, y) = 1 + x^2 + y^2$. The image of Lebesgue measure on sphere under stereographic projection is $4q^{-2}(x, y)dx dy$. Define unbounded operators D_1 and D_2 on (the subspace of all Schwartz functions in) the Hilbert space $L_2(\mathbb{R}^2, 4q^{-2}(x, y)dx dy)$ by setting $D_1 = \frac{1}{i} \frac{\partial}{\partial x}$, $D_2 = \frac{1}{i} \frac{\partial}{\partial y}$. Consider now a couple (A, A^*) of formally adjoint unbounded operators defined on (the subspace of all Schwartz functions in) the Hilbert space $L_2(\mathbb{R}^2, 4q^{-2}(x, y)dx dy)$ by the formula

$$A = \frac{1}{2} M_q (D_1 - iD_2) + \frac{i}{2} M_{\bar{z}}, \quad A^* = \frac{1}{2} M_q (D_1 + iD_2) + \frac{i}{2} M_z.$$

Our Hilbert space is $\mathbb{C}^2 \otimes L_2(\mathbb{R}^2, 4q^{-2}(x, y)dx dy)$. Our von Neumann algebra is $L_\infty(\mathbb{S}^2)$ with a smooth subalgebra $C^\infty(\mathbb{S}^2)$. Its representation is given by the formula $\pi(f) = 1 \otimes M_{f \circ \text{Stereo}^{-1}}$, where Stereo denotes stereographic projection. Our Dirac operator is then defined by the formula (see equation (9.52) in [11])

$$D = e_{12} \otimes A + e_{21} \otimes A^*,$$

where $e_{12}, e_{21} \in M_2(\mathbb{C})$ are matrix units. It is established in Corollary 9.26 and Proposition 9.28 in [11] that D admits an orthonormal eigenbasis. In particular, D is self-adjoint.

By Corollary 9.29 in [11], the constructed spectral triple

$$(\pi(L_\infty(\mathbb{S}^2)), \mathbb{C}^2 \otimes L_2(\mathbb{R}^2, 4q^{-2}(x, y)dx dy), D)$$

is $(2, \infty)$ -summable. In the following lemma, we show that it satisfies the Condition 1.4.

Lemma 4.2. *For every $f \in L_\infty(\mathbb{S}^2)$, we have*

$$\text{Tr}(\pi(f) e^{-t^2 D^2}) = t^{-2} \cdot \frac{1}{4\pi} \int_{\mathbb{S}^2} f(s) ds + O(t^{-1}), \quad t \rightarrow 0.$$

Proof. Recall how the group $SU(2)$ acts on extended complex plane.

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2), \quad g(z) = \frac{az + b}{-bz + \bar{a}}, \quad z \in \mathbb{C}. \quad (4.1)$$

This action results in the unitary representation τ of the group $SU(2)$ on the Hilbert space $\mathbb{C}^2 \otimes L_2(\mathbb{S}^2)$ by the formula

$$\left(\tau(g) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right)(z) = \left(\frac{b\bar{z} + \bar{a}}{\bar{b}z + a} \right)^{\frac{1}{2}} \begin{pmatrix} \psi_1(g^{-1}(z)) \\ \psi_2(g^{-1}(z)) \end{pmatrix}, \quad \psi_1, \psi_2 \in L_2(\mathbb{R}^2, 4q^{-2}(x, y)dx dy).$$

The key fact (Proposition 9.27 in [11]) is that Dirac operator D commutes with the $\tau(g)$ for every $g \in SU(2)$.

Let $f \in L_\infty(\mathbb{S}^2)$ and denote for brevity $F = f \circ \text{Stereo}^{-1}$. Let P be a spectral projection of D . Since P commutes with $\tau(g)$, it follows that

$$\begin{aligned} \text{Tr}((1 \otimes M_F)P) &= \text{Tr}((1 \otimes M_F)\tau(g) \cdot \tau(g^{-1})P) = \text{Tr}((1 \otimes M_F)\tau(g) \cdot P\tau(g^{-1})) = \\ &= \text{Tr}(\tau(g^{-1})(1 \otimes M_F)\tau(g) \cdot P). \end{aligned}$$

Note that

$$\tau(g^{-1})(1 \otimes M_F)\tau(g) = M_{F \circ g}.$$

Thus,

$$\text{Tr}(\pi(f)P) = \text{Tr}((1 \otimes M_F)P) = \text{Tr}((1 \otimes M_{F \circ g})P), \quad g \in SU(2).$$

Since the latter equality holds for every $g \in SU(2)$, it follows that

$$\text{Tr}(\pi(f)P) = \int_{SU(2)} \text{Tr}((1 \otimes M_{F \circ g})P) dg = \text{Tr}((1 \otimes \int_{SU(2)} M_{F \circ g} dg)P),$$

where dg is the Haar measure on $SU(2)$. The action of $SU(2)$ as given in (4.1) is conjugated (by means of stereographic projection, see Section 1.4 in [9]) to the action of $SU(2)$ on sphere \mathbb{S}^2 by rotations. It follows that

$$\int_{SU(2)} (F \circ g) dg = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(s) ds$$

and, therefore,

$$\text{Tr}(\pi(f)P) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(s) ds \cdot \text{Tr}(P).$$

According to Corollary 9.29 in [11], spectrum of D is $\mathbb{Z} \setminus \{0\}$ and, for every $l \in \mathbb{Z} \setminus \{0\}$, $\text{Tr}(E_D\{l\}) = |l|$. It follows that

$$\begin{aligned} \text{Tr}(\pi(f)e^{-t^2 D^2}) &= \sum_{l \in \mathbb{Z} \setminus \{0\}} \text{Tr}(\pi(f)e^{-t^2 D^2} E_D\{l\}) = \\ &= \sum_{l \in \mathbb{Z} \setminus \{0\}} e^{-t^2 l^2} \text{Tr}(\pi(f)E_D\{l\}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(s) ds \cdot \sum_{l \in \mathbb{Z}} |l| e^{-t^2 l^2}. \end{aligned}$$

The assertion follows now from Lemma 4.1 (a). \square

4.2. Example: noncommutative torus

We briefly recall a spectral triple for the noncommutative torus (originally introduced in Section II.2.β in [4]). After that, we show that the triple satisfies the Condition 1.4.

Let $\Theta \in M_p(\mathbb{R})$, $1 < p \in \mathbb{N}$, be an anti-symmetric matrix. Let A_Θ be the universal $*$ -algebra generated by unitaries $\{U_k\}_{k=1}^p$ satisfying the conditions

$$U_{k_2} U_{k_1} = e^{i\Theta_{k_1, k_2}} U_{k_1} U_{k_2}, \quad 1 \leq k_1, k_2 \leq p.$$

Define a linear functional $\tau : A_\Theta \rightarrow \mathbb{C}$ by setting

$$\tau(U_1^{n_1} \cdots U_p^{n_p}) = 0 \text{ unless } (n_1, \dots, n_p) = 0.$$

It can be demonstrated that τ is positive, that is $\tau(x^*x) \geq 0$ for $x \in A_\Theta$. We equip linear space A_Θ with an inner product defined by the formula

$$\langle x, y \rangle = \tau(xy^*), \quad x, y \in A_\Theta.$$

Natural action λ of A_Θ on pre-Hilbert space $(A_\Theta, \langle \cdot, \cdot \rangle)$ by left multiplications extends to the action on the completed Hilbert space. The weak* closure of $\lambda(A_\Theta)$ is denoted by $L_\infty(\mathbb{T}_\Theta^p)$ and τ extends to a faithful normal tracial state on $L_\infty(\mathbb{T}_\Theta^p)$. The Hilbert space where $L_\infty(\mathbb{T}_\Theta^p)$ is naturally identified with $L_2(\mathbb{T}_\Theta^p, \tau)$.

A natural spectral triple for the noncommutative torus is given as follows.⁵ Set $\mathcal{A} = L_\infty(\mathbb{T}_\Theta^p)$ and take $\lambda(A_\Theta)$ to be the subalgebra of smooth elements. Let $m(p) = 2^{\lfloor \frac{p}{2} \rfloor}$. Set $H = \mathbb{C}^{m(p)} \otimes L_2(\mathbb{T}_\Theta^p)$ and $\pi(x) = 1 \otimes M_x$, $x \in L_\infty(\mathbb{T}_\Theta^p)$. Define self-adjoint operators D_k , $1 \leq k \leq p$, on the Hilbert space $L_2(\mathbb{T}_\Theta^p)$ by setting

$$D_k : U_1^{n_1} \cdots U_p^{n_p} \rightarrow n_k U_1^{n_1} \cdots U_p^{n_p}, \quad (n_1, \dots, n_p) \in \mathbb{Z}^p.$$

Those operators commute. Dirac operator D acts on the Hilbert space H by the setting

$$D = \sum_{k=1}^p \gamma_k \otimes D_k,$$

where $\gamma_k \in M_{m(p)}(\mathbb{C})$, $1 \leq k \leq p$, are Pauli matrices.

Lemma 4.3. *For every $x \in L_\infty(\mathbb{T}_\Theta^p)$, we have*

$$\text{Tr}(\pi(x)e^{-t^2 D^2}) = \frac{\pi^{\frac{p}{2}} m(p)}{t^p} \tau(x) + O\left(\frac{1}{t^{p-1}}\right).$$

Proof. Let $\{u_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}}$ be the standard basis in $L_2(\mathbb{T}_\Theta^p, \tau)$, that is

$$u_{\mathbf{k}} = U_1^{k_1} U_2^{k_2} \cdots U_p^{k_p}, \quad \mathbf{k} = (k_1, \dots, k_p) \in \mathbb{Z}^p.$$

⁵The C^* -algebra $\overline{\lambda(A_\Theta)}^{\|\cdot\|_\infty}$ is isomorphic to a universal C^* -algebra constructed by Davidson (see pp.166-170 in [6]).

If $\{e_m\}_{m=1}^{m(p)}$ is the standard unit basis in $\mathbb{C}^{m(p)}$, then the elements $e_m \otimes u_{\mathbf{k}}$, $1 \leq m \leq m(p)$, $\mathbf{k} \in \mathbb{Z}^p$ form an orthonormal basis in $\mathbb{C}^{m(p)} \otimes L_2(\mathbb{T}_{\Theta}^p)$. We have $(D^2)(e_m \otimes u_{\mathbf{k}}) = |\mathbf{k}|^2 e_m \otimes u_{\mathbf{k}}$ and $\langle \pi(x)(e_m \otimes u_{\mathbf{k}}), e_m \otimes u_{\mathbf{k}} \rangle = \tau(x)$ for every $x \in L_{\infty}(\mathbb{T}_{\Theta}^p)$ and for every $1 \leq m \leq p$, $\mathbf{k} \in \mathbb{Z}^p$. Hence,

$$\begin{aligned} \text{Tr}(\pi(x)e^{-t^2 D^2}) &= \sum_{m=1}^{m(p)} \sum_{\mathbf{k} \in \mathbb{Z}^p} \langle \pi(x)e^{-t^2 D^2}(e_m \otimes u_{\mathbf{k}}), e_m \otimes u_{\mathbf{k}} \rangle = \\ &= m(p)\tau(x) \sum_{\mathbf{k} \in \mathbb{Z}^p} e^{-t^2 |\mathbf{k}|^2} = m(p)\tau(x) \left(\sum_{k \in \mathbb{Z}} e^{-t^2 k^2} \right)^p. \end{aligned}$$

The assertion follows now from Lemma 4.1 (b). \square

Proof of Corollary 1.6. By Lemma 4.3, the noncommutative torus satisfies the Condition 1.4. The assertion follows now from Theorem 1.5. \square

4.3. Example: quantum group $SU_q(2)$

In what follows, $\mathcal{O}(SU_q(2))$ is the algebraic linear span of all words in a, c, a^*, c^* with the following cancellation rules⁶

$$a^*a + c^*c = 1, \quad aa^* + q^2cc^* = 1, \quad ac = qca, \quad ac^* = qc^*a, \quad cc^* = c^*c.$$

Here, the parameter q takes value from the interval $[-1, 1]$. For $q = 1$, the algebra $\mathcal{O}(SU_q(2))$ is commutative and (its von Neumann envelope) equals to $L_{\infty}(SU(2))$. In what follows, we assume $q \in (-1, 1)$.

It follows from Proposition IV.4 in [14] that the elements

$$\{a^n c^m (c^*)^r, c^m (c^*)^r (a^*)^{n+1}\}_{m,n,r \in \mathbb{Z}_+}$$

form a Hamel basis in $\mathcal{O}(SU_q(2))$. Define a linear functional⁷ τ on $\mathcal{O}(SU_q(2))$ by setting

$$\tau(a^n c^m (c^*)^r) = \tau(c^m (c^*)^r (a^*)^n) = 0, \quad (m, n, r) \neq (0, 0, 0), \quad \tau(1) = 1.$$

The algebra $\mathcal{O}(SU_q(2))$ acts on the Hilbert space H which is the completion of $\mathcal{O}(SU_q(2))$ with respect to the inner product $(x, y) \rightarrow h(xy^*)$. Here, h is the Haar state on $SU_q(2)$ (defined in Theorem IV.14 in [14]).

Let $l, m, n \in \frac{1}{2}\mathbb{Z}_+$ be such that $l \geq 0$, $|m|, |n| \leq l$ and $l - m, l - n \in \mathbb{Z}$. Let $t_{m,n}^l$ be the orthonormal basis in $\mathcal{O}(SU_q(2))$ constructed in Theorem IV.13 in [14]. By construction of the Hilbert space H , these elements also form an

⁶Here, we are using the notations from [14]. The definition appears on p.102 in [14]. The same definition is used by Connes (see formula (18) in [5]) and Chakraborty-Pal (see p.2 in [3]). However, those authors use the notation α for a and β for c .

⁷This is a trace on $\mathcal{O}(SU_q(2))$, not the Haar state. However, we don't need the tracial property of τ . Connes (see Theorem 4 in [5]) considered this functional on a larger algebra.

orthonormal basis in H . Connes defined Dirac operator D on the Hilbert space H in [5] (see formula (22) on p.8 there) by the formula

$$Dt_{m,n}^l = 2l(2\delta_0(l-m) - 1)t_{m,n}^l.$$

In this text, we are not interested in the sign of D , but only in its absolute value given by the formula

$$|D|t_{m,n}^l = 2lt_{m,n}^l.$$

Lemma 4.4. *For every $x \in \mathcal{O}(SU_q(2))$, we have*

$$\mathrm{Tr}(M_x e^{-(tD)^2}) = \frac{\pi^{\frac{1}{2}}}{4t^3} \tau(x) + O(t^{-2}), \quad t \rightarrow 0.$$

Proof. Step 1: Let $j \in \frac{1}{2}\mathbb{Z}_+$ and let $(r, s) \neq (0, 0)$. We claim that

$$\mathrm{Tr}(M_{t_{r,s}^j} e^{-(tD)^2}) = 0.$$

Let $\mathcal{A}[\cdot, \cdot]$ be the linear subspace in $\mathcal{O}(SU_q(2))$ defined on p.105 in [14]. By Lemma IV.11 in [14], we have $t_{r,s}^j \in \mathcal{A}[-2r, -2s]$ and $t_{m,n}^l \in \mathcal{A}[-2m, -2n]$. Using formula (26) on p.105 in [14], we infer that $t_{r,s}^j t_{m,n}^l \in \mathcal{A}[-2r-2m, -2s-2n]$. It follows now from formulae (47) and (56) in [14] that $t_{r,s}^j t_{m,n}^l$ is orthogonal to $t_{m,n}^l$ (because $(r, s) \neq (0, 0)$). Thus,

$$\mathrm{Tr}(M_{t_{r,s}^j} e^{-(tD)^2}) = \sum_{l,m,n} e^{-4t^2 l^2} \langle t_{r,s}^j t_{m,n}^l, t_{m,n}^l \rangle = 0.$$

Step 2: Let $j \in \mathbb{Z}_+$. We claim that

$$\mathrm{Tr}(M_{(cc^*)^j} e^{-(tD)^2}) = O(t^{-2}).$$

In what follows, $p_l = E_{|D|}\{2l\}$. Using formulae (2.1)–(2.9) in [3] (or formulae (19)–(21) in [5]), we infer that

$$(p_l M_{cc^*} p_l) t_{m,n}^l = c(m, n, l) t_{m,n}^l,$$

where

$$c(m, n, l) = O(q^{m+l}) + O(q^{n+l}).$$

Taking into account that $|q| < 1$, we obtain

$$\mathrm{Tr}(p_l M_{cc^*} p_l) = \sum_{m,n=-l}^l c(m, n, l) = \sum_{m,n=-l}^l \left(O(q^{m+l}) + O(q^{n+l}) \right) = O(l).$$

It follows that

$$\mathrm{Tr}(M_{(cc^*)^j} p_l) = \mathrm{Tr}(p_l M_{(cc^*)^j} p_l) \leq \|b\|_\infty^{2j-2} \mathrm{Tr}(p_l M_{cc^*} p_l) = O(l).$$

Hence,

$$\mathrm{Tr}(M_{(cc^*)^j} e^{-(tD)^2}) = \sum_{l \in \frac{1}{2}\mathbb{Z}_+} e^{-4t^2 l^2} \mathrm{Tr}(M_{(cc^*)^j} p_l) = \sum_{l \in \frac{1}{2}\mathbb{Z}_+} e^{-4t^2 l^2} \cdot O(l).$$

Thus,

$$|\mathrm{Tr}(M_{(cc^*)^j} e^{-(tD)^2})| \leq O(1) \cdot \left(\sum_{l \in \frac{1}{2}\mathbb{Z}_+} l e^{-4l^2 t^2} \right) = O(1) \cdot \left(\sum_{l \in \mathbb{Z}_+} l e^{-l^2 t^2} \right).$$

Replacing the sum with an integral, we conclude the proof in Step 2.

Step 3: Let $r, s \in \frac{1}{2}\mathbb{Z}$ be such that $(r, s) \neq (0, 0)$. By Step 1 and formula (47) in [14], we have

$$\mathrm{Tr}(M_x e^{-(tD)^2}) = 0, \quad x \in \mathcal{A}[-2r, -2s].$$

By Step 2 and Proposition 10 (i) in [14], we have

$$\mathrm{Tr}(M_x e^{-(tD)^2}) = O(t^{-2}), \quad x \in \mathcal{A}[0, 0], \quad \tau(x) = 0.$$

A trivial computation shows that

$$\mathrm{Tr}(e^{-(tD)^2}) = \sum_{l \in \frac{1}{2}\mathbb{Z}_+} (2l+1)^2 e^{-4l^2 t^2} = O(t^{-2}) + \frac{1}{2} \sum_{l \in \mathbb{Z}} l^2 e^{-l^2 t^2}.$$

It follows from Lemma 4.1 (c) that

$$\mathrm{Tr}(e^{-(tD)^2}) = \frac{\pi^{\frac{1}{2}}}{4t^3} + O(t^{-2}).$$

Thus, for every $m, n \in \mathbb{Z}$, we have

$$\mathrm{Tr}(M_x e^{-(tD)^2}) = \frac{\pi^{\frac{1}{2}}}{4t^3} \tau(x) + O(t^{-2}), \quad x \in \mathcal{A}[m, n].$$

The assertion follows now from formula (32) in [14]. \square

Proof of Corollary 1.7. By Lemma 4.2 and Lemma 4.4, spectral triples corresponding to the sphere \mathbb{S}^2 and to the quantum group $\mathrm{SU}_q(2)$ satisfy the Condition 1.4. The assertion follows now from Theorem 1.5. \square

5. Proof of Theorem 1.12

The following lemma provides a convenient formula for the sum of the first n eigenvalues of $\mathrm{diag}(x) \in \mathcal{L}_{1,\infty}$.

Lemma 5.1. *If $x \in l_\infty$ is such that $|x(k)| \leq \frac{1}{k+1}$, $k \geq 0$, then*

$$\sum_{m=0}^n \lambda(m, x) = \sum_{k=0}^n x(k) + O(1).$$

Proof. Suppose first that $x \geq 0$. It is clear that

$$\sum_{k=0}^n x(k) \leq \sum_{k=0}^n \mu(k, x).$$

On the other hand, there exists a set $A_n \subset \mathbb{Z}_+$ such that $|A| = n + 1$ and such that

$$\begin{aligned} \sum_{k=0}^n \mu(k, x) &= \sum_{k \in A_n} x(k) = \sum_{k \in A_n, k \leq n} x(k) + \sum_{k \in A_n, k \geq n} x(k) \leq \\ &\leq \sum_{k=0}^n x(k) + \sum_{k \in A_n, k \geq n} \frac{1}{k+1} \leq \sum_{k=0}^n x(k) + \sum_{k=n+1}^{2n+1} \frac{1}{k+1} \leq \sum_{k=0}^n x(k) + 1. \end{aligned}$$

A combination of the latter estimates yields the assertion under the additional assumption that $x \geq 0$.

For an arbitrary $x \in l_{1,\infty}$, there exist $0 \leq x_p \in l_{1,\infty}$, $1 \leq p \leq 4$, such that

$$x = x_1 + ix_2 + i^2x_3 + i^3x_4.$$

If $|x(k)| \leq \frac{1}{k+1}$, $k \geq 0$, then also $x_p(k) \leq \frac{1}{k+1}$, $k \geq 0$. It follows from Lemma 5.7.5 in [16] that

$$\sum_{m=0}^n \lambda(m, x) = \sum_{p=1}^4 i^{p-1} \sum_{m=0}^n \lambda(m, x_p) + O(1).$$

Applying the assertion for positive operators x_p , $1 \leq p \leq 4$, we infer that

$$\sum_{m=0}^n \lambda(m, x) = \sum_{p=1}^4 i^{p-1} \sum_{k=0}^n x_p(k) + O(1) = \sum_{k=0}^n x(k) + O(1).$$

This concludes the proof. \square

Lemma 5.2. *If $x \in l_\infty(\mathbb{Z}^2)$ is such that $|x(k, l)| \leq \frac{1}{1+k^2+l^2}$, $k, l \geq 0$, then*

$$\sum_{m=0}^n \lambda(m, x) = \sum_{k,l=0}^{\lfloor n^{1/2} \rfloor} x(k, l) + O(1).$$

Proof. Define a bijection $\alpha_2 : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+^2$ as in Lemma Appendix A.2. Define $z \in l_\infty$ by setting $z = x \circ \alpha_2$. It follows from Lemma Appendix A.2 that

$$|z(m)| \leq \frac{1}{1 + |\alpha_2(m)|^2} \leq \frac{\text{const}}{m+1}, \quad m \geq 0.$$

Therefore,

$$\sum_{m=0}^n \lambda(m, x) = \sum_{m=0}^n \lambda(m, z) \stackrel{L.5.1}{=} \sum_{m=0}^n z(m) + O(1) = \sum_{m=0}^n x(\alpha_2(m)) + O(1).$$

Note that

$$\begin{aligned} \left| \sum_{m=0}^n x(\alpha_2(m)) - \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^2 \\ |\mathbf{k}| \leq |\alpha_2(n)|}} x(\mathbf{k}) \right| &\leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^2 \\ |\mathbf{k}| = |\alpha_2(n)|}} |x(\mathbf{k})| \leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^2 \\ |\mathbf{k}| = |\alpha_2(n)|}} \frac{1}{1 + |\mathbf{k}|^2} = \\ &= \frac{1}{1 + |\alpha_2(n)|^2} \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^2 \\ |\mathbf{k}| = |\alpha_2(n)|}} 1 \leq \frac{1}{1 + |\alpha_2(n)|^2} \sum_{k=0}^{|\alpha_2(n)|} 1 \leq 1. \end{aligned}$$

It follows from Lemma Appendix A.2 that

$$\left| \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^2 \\ |\mathbf{k}| \leq |\alpha_2(n)|}} x(\mathbf{k}) - \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^2 \\ |\mathbf{k}|^2 \leq n}} x(\mathbf{k}) \right| \leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^2 \\ |\mathbf{k}|^2 \in [|\alpha_2(n)|^2, n]}} |x(\mathbf{k})| \leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^2 \\ |\mathbf{k}|^2 \in [c_1 n, c_2 n]}} \frac{1}{1 + |\mathbf{k}|^2} = O(1).$$

We also have

$$\begin{aligned} \left| \sum_{\substack{k_1^2 + k_2^2 \leq n}} x(k_1, k_2) - \sum_{0 \leq k_1, k_2 \leq n^{1/2}} x(k_1, k_2) \right| &\leq \sum_{\substack{0 \leq k_1, k_2 \leq n^{1/2} \\ k_1^2 + k_2^2 > n}} |x(k_1, k_2)| \leq \\ &\leq \sum_{\substack{0 \leq k_1, k_2 \leq n^{1/2} \\ k_1^2 + k_2^2 > n}} \frac{1}{1 + k_1^2 + k_2^2} \leq \frac{1}{1 + n} \sum_{0 \leq k_1, k_2 \leq n^{1/2}} 1 = O(1). \end{aligned}$$

A combination of the latter estimates yields the assertion. \square

For a given $\theta \in \mathbb{R}$, we define $x_\theta \in l_\infty$ by setting $x_\theta(0) = 1$ and

$$x_\theta(k) = e^{in\theta}, \quad k \in [2^n, 2^{n+1}), \quad n \geq 0.$$

Let $D = \text{diag}(\{k\}_{k \geq 0})$, $U_\theta = \text{diag}(\{x_\theta(k)\}_{k \geq 0})$. Clearly, $(1 + D^2)^{-1/2} \in \mathcal{L}_{1,\infty}$.

Lemma 5.3. *For every $\theta \notin 2\pi\mathbb{Z}$, we have $U_\theta D^{-1} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$.*

Proof. We have

$$\begin{aligned} \sum_{k=1}^{2^{n+1}-1} \frac{x_\theta(k)}{k} &= \sum_{m=0}^n e^{im\theta} \sum_{k=2^m}^{2^{m+1}-1} \frac{1}{k} = \sum_{m=0}^n e^{im\theta} (\log(2) + O(2^{-m})) = \\ &= O(1) + \log(2) \frac{e^{i(n+1)\theta} - e^{i\theta}}{e^{i\theta} - 1} = O(1). \end{aligned}$$

Hence, for every $m \geq 1$, we have

$$\sum_{k=0}^m \frac{x_\theta(k)}{(1 + k^2)^{1/2}} = O(1).$$

By Lemma 5.1, we have

$$\sum_{k=0}^m \lambda(U_\theta(1 + D^2)^{-1/2}) = O(1).$$

The assertion follows from Theorem 2.3. \square

Lemma 5.4. *For every $m_1, m_2 \in \mathbb{Z}_+$, we have*

$$\sum_{k=2^{m_1}}^{2^{m_1+1}-1} \sum_{l=2^{m_2}}^{2^{m_2+1}-1} \frac{1}{k^2 + l^2} = \Xi(|m_1 - m_2|) + O(\min\{2^{-m_1}, 2^{-m_2}\}).$$

Here,

$$\Xi(m) = \int_1^2 \int_{2^m}^{2^{m+1}} \frac{dt ds}{t^2 + s^2}, \quad m \in \mathbb{Z}. \quad (5.1)$$

Proof. It is clear that

$$\int_{2^{m_2}}^{2^{m_2+1}} \frac{dt}{t^2 + k^2} \leq \sum_{l=2^{m_2}}^{2^{m_2+1}-1} \frac{1}{k^2 + l^2} \leq \int_{2^{m_2}}^{2^{m_2+1}} \frac{dt}{t^2 + k^2} + \frac{1}{k^2 + 2^{2m_2}}.$$

Thus,

$$\sum_{l=2^{m_2}}^{2^{m_2+1}-1} \frac{1}{k^2 + l^2} = \int_{2^{m_2}}^{2^{m_2+1}} \frac{dt}{t^2 + k^2} + \frac{O(1)}{k^2 + 2^{2m_2}}.$$

It follows that

$$\sum_{k=2^{m_1}}^{2^{m_1+1}-1} \sum_{l=2^{m_2}}^{2^{m_2+1}-1} \frac{1}{k^2 + l^2} = \int_{2^{m_2}}^{2^{m_2+1}} \sum_{k=2^{m_1}}^{2^{m_1+1}-1} \frac{dt}{t^2 + k^2} + O(1) \cdot \sum_{k=2^{m_1}}^{2^{m_1+1}-1} \frac{1}{k^2 + 2^{2m_2}}.$$

Repeating the argument, we obtain that

$$\begin{aligned} & \sum_{k=2^{m_1}}^{2^{m_1+1}-1} \sum_{l=2^{m_2}}^{2^{m_2+1}-1} \frac{1}{k^2 + l^2} = \int_{2^{m_2}}^{2^{m_2+1}} \int_{2^{m_1}}^{2^{m_1+1}} \frac{dt ds}{t^2 + s^2} + \\ & + O(1) \cdot \int_{2^{m_2}}^{2^{m_2+1}} \frac{dt}{t^2 + 2^{2m_1}} + O(1) \cdot \int_{2^{m_1}}^{2^{m_1+1}} \frac{ds}{s^2 + 2^{2m_2}} + \frac{O(1)}{2^{2m_1} + 2^{2m_2}}. \end{aligned}$$

Clearly, the second and third integrals above can be estimated as

$$O(1) \cdot \frac{2^{m_2}}{2^{2m_1} + 2^{2m_2}}, \quad O(1) \cdot \frac{2^{m_1}}{2^{2m_1} + 2^{2m_2}}.$$

The reference to (5.1) concludes the proof. \square

It is obvious that

$$0 \leq \Xi(m) \leq 2^{-m}, \quad m \in \mathbb{Z}_+. \quad (5.2)$$

Lemma 5.5. *For every $\theta \in \mathbb{R}$ and for every $p \in \mathbb{Z}$, we have*

$$\sum_{k,l=1}^M \frac{x_\theta^p(k)x_\theta^{-p}(l)}{k^2+l^2} = F(p\theta) \frac{\log(M)}{\log(2)} + O(1), \quad M \in \mathbb{N}.$$

Here,

$$F(\theta) = \sum_{m \in \mathbb{Z}} \Xi(|m|) e^{im\theta}, \quad (5.3)$$

where Ξ is given in (5.1).

Proof. Since $x_\theta^p = x_{p\theta}$, it follows that we may consider only the case $p = 1$. Firstly, we establish the assertion for $M = 2^{n+1} - 1$, $n \in \mathbb{Z}_+$. It follows from Lemma 5.4 that

$$\begin{aligned} \sum_{k,l=1}^{2^{n+1}-1} \frac{x_\theta(k)x_\theta^{-1}(l)}{k^2+l^2} &= \sum_{m_1, m_2=0}^n e^{i(m_1-m_2)\theta} \sum_{k=2^{m_1}}^{2^{m_1+1}-1} \sum_{l=2^{m_2}}^{2^{m_2+1}-1} \frac{1}{k^2+l^2} = \\ &= \sum_{m_1, m_2=0}^n \Xi(|m_1-m_2|) e^{i(m_1-m_2)\theta} + O(1). \end{aligned}$$

Rearranging the summands, we obtain that

$$\begin{aligned} &\sum_{m_1, m_2=0}^n \Xi(|m_1-m_2|) e^{i(m_1-m_2)\theta} = \\ &= (n+1)\Xi(0) + \sum_{m=1}^n (n+1-m) e^{im\theta} \Xi(m) + \sum_{m=1}^n (n+1-m) e^{-im\theta} \Xi(m). \end{aligned}$$

It follows from (5.2) that

$$\sum_{m_1, m_2=0}^n \Xi(|m_1-m_2|) e^{i(m_1-m_2)\theta} = n \left(\sum_{m=-n}^n \Xi(|m|) e^{im\theta} \right) + O(1) = nF(\theta) + O(1).$$

This proves the assertion for $M = 2^{n+1} - 1$, $n \in \mathbb{Z}_+$.

Now, for an arbitrary $M \in [2^n, 2^{n+1})$, we have

$$\begin{aligned} \left| \sum_{k,l=1}^{2^{n+1}-1} \frac{x_\theta(k)x_\theta^{-1}(l)}{k^2+l^2} - \sum_{k,l=1}^M \frac{x_\theta(k)x_\theta^{-1}(l)}{k^2+l^2} \right| &\leq \sum_{k,l=1}^{2^{n+1}-1} \frac{1}{k^2+l^2} - \sum_{k,l=1}^{2^n-1} \frac{1}{k^2+l^2} \leq \\ &\leq 2 \sum_{k=2^n}^{2^{n+1}-1} \sum_{l=1}^{2^{n+1}-1} \frac{1}{k^2+l^2} \leq 2 \sum_{k=2^n}^{2^{n+1}-1} \sum_{l=1}^{2^{n+1}-1} 2^{-2n} \leq 4. \end{aligned}$$

This concludes the proof. \square

The proof of the following lemma is parallel (though, not identical) to that of Lemma 5.5.

Lemma 5.6. *For every $\theta \in \mathbb{R}$ and for every $p, q \in \mathbb{Z}$ such that $(p+q)\theta \notin 2\pi\mathbb{Z}$, we have*

$$\sum_{k,l=1}^M \frac{x_\theta^p(k)x_\theta^q(l)}{k^2+l^2} = O(1), \quad M \in \mathbb{N}.$$

Proof. Firstly, we establish the assertion for $M = 2^{n+1} - 1$, $n \in \mathbb{Z}_+$. It follows from Lemma 5.4 that

$$\begin{aligned} \sum_{k,l=1}^{2^{n+1}-1} \frac{x_\theta^p(k)x_\theta^q(l)}{k^2+l^2} &= \sum_{m_1, m_2=0}^n e^{i(p m_1 + q m_2)\theta} \sum_{k=2^{m_1}}^{2^{m_1+1}-1} \sum_{l=2^{m_2}}^{2^{m_2+1}-1} \frac{1}{k^2+l^2} = \\ &= \sum_{m_1, m_2=0}^n \Xi(|m_1 - m_2|) e^{ip(m_1-m_2)\theta} e^{i(p+q)m_2\theta} + O(1). \end{aligned}$$

Rearranging the summands, we obtain that

$$\begin{aligned} \sum_{m_1, m_2=0}^n \Xi(|m_1 - m_2|) e^{ip(m_1-m_2)\theta} e^{i(p+q)m_2\theta} &= \sum_{m=0}^n \Xi(0) e^{i(p+q)m\theta} + \\ &+ \sum_{m=1}^n \Xi(m) e^{ipm\theta} \sum_{m_2=0}^{n-m} e^{i(p+q)m_2\theta} + \sum_{m=1}^n \Xi(m) e^{-ipm\theta} \sum_{m_2=m}^n e^{i(p+q)m_2\theta}. \end{aligned}$$

The assumption $(p+q)\theta \notin 2\pi\mathbb{Z}$ guarantees that

$$\sum_{m_2=0}^{n-m} e^{i(p+q)m_2\theta} = O(1), \quad \sum_{m_2=m}^n e^{i(p+q)m_2\theta} = O(1), \quad m, n \in \mathbb{Z}_+.$$

Therefore, appealing to (5.2), we obtain

$$\begin{aligned} \left| \sum_{m_1, m_2=0}^n \Xi(|m_1 - m_2|) e^{ip(m_1-m_2)\theta} e^{i(p+q)m_2\theta} \right| &\leq \left| \sum_{m=0}^n \Xi(0) e^{i(p+q)m\theta} \right| + \\ &+ \sum_{m=1}^n \Xi(m) \cdot O(1) + \sum_{m=1}^n \Xi(m) \cdot O(1) = O(1). \end{aligned}$$

In other words, we have

$$\sum_{k,l=1}^{2^{n+1}-1} \frac{x_\theta^p(k)x_\theta^q(l)}{k^2+l^2} = O(1).$$

This proves the assertion for $M = 2^{n+1} - 1$, $n \in \mathbb{Z}_+$.

Now, for an arbitrary $M \in [2^n, 2^{n+1})$, we have

$$\begin{aligned} \left| \sum_{k,l=1}^{2^{n+1}-1} \frac{x_\theta^p(k)x_\theta^q(l)}{k^2+l^2} - \sum_{k,l=1}^M \frac{x_\theta^p(k)x_\theta^q(l)}{k^2+l^2} \right| &\leq \sum_{k,l=1}^{2^{n+1}-1} \frac{1}{k^2+l^2} - \sum_{k,l=1}^{2^n-1} \frac{1}{k^2+l^2} \leq \\ &\leq 2 \sum_{k=2^n}^{2^{n+1}-1} \sum_{l=1}^{2^{n+1}-1} \frac{1}{k^2+l^2} \leq 2 \sum_{k=2^n}^{2^{n+1}-1} \sum_{l=1}^{2^{n+1}-1} 2^{-2n} \leq 4. \end{aligned}$$

This concludes the proof. \square

Proof of Theorem 1.12. Let F be as in (5.3). Fourier coefficients of F are given by a non-zero sequence $\{\Xi(|m|)\}_{m \in \mathbb{Z}}$ and, therefore $F \neq 0$. It follows from (5.2) that Fourier series for F converges uniformly and, therefore, F is continuous. It follows from the continuity of F that one can choose θ such that $\frac{\theta}{2\pi} \in \mathbb{Q}$, $\theta \notin 2\pi\mathbb{Z}$ and such that $F(\theta) \neq 0$. Let \mathcal{A}_θ be the von Neumann subalgebra in $\mathcal{L}(l_2)$ generated by U_θ .

Since $\theta \in 2\pi\mathbb{Q}$, it follows that there exists $0 \neq r \in \mathbb{Z}$ such that $U_\theta^r = 1$ and, therefore, \mathcal{A}_θ is finite dimensional. Every linear functional on a finite dimensional subalgebra in $\mathcal{L}(H)$ is automatically normal. It follows that the mapping

$$T \rightarrow \varphi(T(1+D^2)^{-1/2}), \quad T \in \mathcal{A}_\theta$$

is normal for every linear functional on $\mathcal{L}_{1,\infty}$ (in particular, for every trace on $\mathcal{L}_{1,\infty}$). It follows from Lemma 5.3 that, for every $p \in \mathbb{Z}$ and for every normalised trace φ on $\mathcal{L}_{1,\infty}$, we have

$$\varphi(U_\theta^p(1+D^2)^{-1/2}) = \begin{cases} 1, & p\theta \in 2\pi\mathbb{Z} \\ 0, & p\theta \notin 2\pi\mathbb{Z}. \end{cases}$$

Hence, $T(1+D^2)^{-1/2}$ is universally measurable for every $T \in \mathcal{A}_\theta$. This proves (c).

Since $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ is also finite dimensional, it follows that the mapping

$$T \rightarrow \varphi(T(1+D^2 \otimes 1 + 1 \otimes D^2)^{-1/2}), \quad T \in \mathcal{A}_\theta \otimes \mathcal{A}_\theta$$

is automatically normal for every linear functional on $\mathcal{L}_{1,\infty}$ (in particular, for every trace on $\mathcal{L}_{1,\infty}$).

It follows from Lemma 5.5 that, for every $\theta \in \mathbb{R}$ and for every $p \in \mathbb{Z}$, we have

$$\sum_{k,l=0}^M \frac{x_\theta^p(k)x_\theta^{-p}(l)}{1+k^2+l^2} = F(p\theta) \frac{\log(M+1)}{\log(2)} + O(1), \quad M \in \mathbb{Z}_+.$$

This equality combined with Lemma 5.2 provides that

$$\sum_{m=0}^N \lambda(m, (U_\theta^p \otimes U_\theta^{-p})(1+1 \otimes D^2 + D^2 \otimes 1)^{-1}) = F(p\theta) \frac{\log(N+1)}{2 \log(2)} + O(1), \quad N \in \mathbb{Z}.$$

By Theorem 2.3, we have that

$$\varphi((U_\theta^p \otimes U_\theta^{-p})(1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) = \frac{1}{2\log(2)} F(p\theta) \quad (5.4)$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

It follows from Lemma 5.6 that, for every $\theta \in \mathbb{R}$ and for every $p, q \in \mathbb{Z}$ such that $(p+q)\theta \notin 2\pi\mathbb{Z}$, we have

$$\sum_{k,l=0}^M \frac{x_\theta^p(k)x_\theta^q(l)}{1+k^2+l^2} = O(1), \quad M \in \mathbb{Z}_+.$$

By Lemma 5.2, we have that

$$\sum_{m=0}^N \lambda(m, (U_\theta^p \otimes U_\theta^q)(1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) = O(1), \quad N \in \mathbb{Z}$$

for every $p, q \in \mathbb{Z}$ with $(p+q)\theta \notin 2\pi\mathbb{Z}$. By Theorem 2.3, we have

$$\varphi((U_\theta^p \otimes U_\theta^q)(1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) = 0 \quad (5.5)$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

Combining (5.4) and (5.5), we conclude that elements of the form

$$T(1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}, \quad T \in \mathcal{A}_\theta \otimes \mathcal{A}_\theta,$$

are universally measurable. This proves (d).

Finally, the first assertion in (e) follows from (5.4) (for $p = 1$) and the second assertion in (e) follows from Lemma 5.3. \square

Proof of Corollary (1.13). Set $T = U + U^{-1} + 2 \geq 0$. In the course of the proof of Theorem 1.12, we established a formula (5.5), which implies

$$\varphi((U \otimes U)(1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) = \varphi((U^{-1} \otimes U^{-1})(1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) = 0.$$

It follows from Theorem 1.12 (e) that

$$\begin{aligned} \varphi((T \otimes T)(1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) &= 4\varphi((1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) + \\ &+ 2\varphi((U \otimes U^{-1})(1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) \neq 4\varphi((1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}). \end{aligned}$$

On the other hand, it follows from Lemma Appendix A.3 and Theorem 1.12 (e) that

$$4\varphi((1 + 1 \otimes D^2 + D^2 \otimes 1)^{-1}) = \pi = \frac{\pi}{4}(\varphi(T(1 + D^2)^{-1/2}))^2.$$

\square

Remark 5.7. Neither Theorem 1.12 nor its proof specifies the dimension of the algebra \mathcal{A} . However, if we replace 2 with 2^7 in the definition of x_θ and set $\theta = \pi$, then the algebra \mathcal{A} becomes 2-dimensional. That $F(\pi) \neq 0$ can be showed as in the proof of Theorem 1.14 below.

6. Proof of Theorem 1.14

Define the sequence d by setting $d(0) = 0$,

$$d(k) = \begin{cases} k, & k \in [2^{7n}, 2^{7(n+1)}), \quad n = 0 \bmod 2 \\ 2^7 k, & k \in [2^{7n}, 2^{7(n+1)}), \quad n = 1 \bmod 2 \end{cases}$$

and set $D = \text{diag}(\{d(k)\}_{k \geq 0})$.

Set

$$\Xi_0(m) = \int_1^{2^7} \int_{2^{7m}}^{2^{7(m+1)}} \frac{dt ds}{t^2 + s^2}.$$

The proof of the following lemma is identical to that of Lemma 5.4 and is, therefore, omitted.

Lemma 6.1. *We have*

$$\begin{aligned} & \sum_{k_1=2^{7n_1}}^{2^{7(n_1+1)}-1} \sum_{k_2=2^{7n_2}}^{2^{7(n_2+1)}-1} \frac{1}{d^2(k_1) + d^2(k_2)} = \\ & = O(\min\{2^{-7n_1}, 2^{-7n_2}\}) + \begin{cases} \Xi_0(n_2 - n_1), & n_1 = 0 \bmod 2, n_2 = 0 \bmod 2 \\ 2^{-7}\Xi_0(n_2 - n_1 - 1), & n_1 = 1 \bmod 2, n_2 = 0 \bmod 2 \\ 2^{-7}\Xi_0(n_2 - n_1 + 1), & n_1 = 0 \bmod 2, n_2 = 1 \bmod 2 \\ 2^{-14}\Xi_0(n_2 - n_1), & n_1 = 1 \bmod 2, n_2 = 1 \bmod 2 \end{cases} \end{aligned}$$

Note that

$$\Xi_0(m) = \Xi_0(|m|) \leq \int_1^{2^7} \int_{2^{7|m|}}^{2^{7(|m|+1)}} \frac{dt ds}{2^{14|m|}} \leq (2^7 - 1)^2 \cdot 2^{-7m}.$$

In particular, we have

$$\sum_{m \in \mathbb{Z}} \Xi_0(m) < \infty.$$

Lemma 6.2. *We have*

$$\sum_{k_1, k_2=1}^M \frac{1}{d^2(k_1) + d^2(k_2)} = \frac{\log(M)}{14 \log(2)} (1 + 2^{-7})^2 \sum_{m \in \mathbb{Z}} \Xi_0(2m) + O(1).$$

Proof. Suppose first that $M = 2^{7(n+1)} - 1$. It follows from Lemma 6.1 that

$$\begin{aligned} & \sum_{k_1, k_2=1}^{2^{7(n+1)}-1} \frac{1}{d^2(k_1) + d^2(k_2)} = \sum_{\substack{0 \leq n_1, n_2 \leq n \\ n_1, n_2 = 0 \bmod 2}} \Xi_0(n_2 - n_1) + 2^{-14} \sum_{\substack{0 \leq n_1, n_2 \leq n \\ n_1, n_2 = 1 \bmod 2}} \Xi_0(n_2 - n_1) + \\ & + 2^{-7} \sum_{\substack{0 \leq n_1, n_2 \leq n \\ n_1 = 1 \bmod 2 \\ n_2 = 0 \bmod 2}} \Xi_0(n_2 - n_1 - 1) + 2^{-7} \sum_{\substack{0 \leq n_1, n_2 \leq n \\ n_1 = 0 \bmod 2 \\ n_2 = 1 \bmod 2}} \Xi_0(n_2 - n_1 + 1) + O(1). \end{aligned}$$

Making the substitution

$$(n_1, n_2) = \begin{cases} (m_1, m_2), & n_1 = 0 \bmod 2, n_2 = 0 \bmod 2 \\ (m_1 - 1, m_2), & n_1 = 1 \bmod 2, n_2 = 0 \bmod 2 \\ (m_1, m_2 - 1), & n_1 = 0 \bmod 2, n_2 = 1 \bmod 2 \\ (m_1 - 1, m_2 - 1), & n_1 = 1 \bmod 2, n_2 = 1 \bmod 2 \end{cases}$$

we have that

$$\sum_{k_1, k_2=1}^{2^{7(n+1)}-1} \frac{1}{d^2(k_1) + d^2(k_2)} = (1 + 2^{-7})^2 \sum_{\substack{0 \leq m_1 \leq n \\ 1 \leq m_2 \leq n \\ m_1, m_2 = 0 \bmod 2}} \Xi_0(m_2 - m_1) + O(1).$$

Rearranging the summands as in Lemma 5.5, we infer that

$$\sum_{\substack{0 \leq m_1 \leq n \\ 1 \leq m_2 \leq n \\ m_1, m_2 = 0 \bmod 2}} \Xi_0(m_2 - m_1) = \frac{n}{2} \sum_{m \in \mathbb{Z}} \Xi_0(2m) + O(1).$$

Passing from $M = 2^{7(n+1)} - 1$ to generic M as in Lemma 5.5, we conclude the proof. \square

Lemma 6.3. *For every normalised trace φ on $\mathcal{L}_{1,\infty}$, we have*

$$\begin{aligned} \varphi((1 + D^2 \otimes 1 + 1 \otimes D^2)^{-1}) &= \frac{1}{7 \log(2)} \left(\frac{1 + 2^{-7}}{2} \right)^2 \sum_{m \in \mathbb{Z}} \Xi_0(2m), \\ \varphi((1 + D^2)^{-1/2}) &= \frac{1 + 2^{-7}}{2}. \end{aligned}$$

Proof. It follows from Lemma 6.1 that

$$\sum_{k_1, k_2=0}^M \frac{1}{1 + d^2(k_1) + d^2(k_2)} = \frac{\log(M)}{14 \log(2)} (1 + 2^{-7})^2 \sum_{m \in \mathbb{Z}} \Xi_0(2m) + O(1).$$

It follows now from Lemma 5.2 that

$$\sum_{k=0}^M \lambda(k, (1 + D^2 \otimes 1 + 1 \otimes D^2)^{-1}) = \frac{\log(M)}{28 \log(2)} (1 + 2^{-7})^2 \sum_{m \in \mathbb{Z}} \Xi_0(2m) + O(1).$$

The first assertion follows now from Theorem 2.3.

The second assertion follows from the equality

$$\begin{aligned} \frac{1}{(1 + d^2(k))^{1/2}} &= O(k^{-2}) + \begin{cases} k^{-1}, & k \in [2^{7n}, 2^{7(n+1)}), \quad n = 0 \bmod 2 \\ 2^{-7}k^{-1}, & k \in [2^{7n}, 2^{7(n+1)}), \quad n = 1 \bmod 2 \end{cases} = \\ &= O(k^{-2}) + \frac{1 + 2^{-7}}{2k} + \frac{1 - 2^{-7}}{2k} \cdot \begin{cases} 1, & k \in [2^{7n}, 2^{7(n+1)}), \quad n = 0 \bmod 2 \\ -1, & k \in [2^{7n}, 2^{7(n+1)}), \quad n = 1 \bmod 2. \end{cases} \end{aligned}$$

\square

Proof of Theorem 1.14. According to the Lemma 6.3, it suffices to show that

$$\sum_{m \in \mathbb{Z}} \Xi_0(2m) > \frac{7\pi}{4} \log(2).$$

In fact, we have

$$\begin{aligned} \Xi_0(0) &= \int_1^{2^7} \int_1^{2^7} \frac{dt ds}{t^2 + s^2} = 2 \int_{1 \leq s \leq t \leq 2^7} \frac{dt ds}{t^2 + s^2} \geq 2 \int_{1 \leq s \leq t \leq 2^7} \frac{dt ds}{2t^2} = \\ &= \int_1^{2^7} \frac{(t-1)dt}{t^2} > 7 \log(2) - 1. \end{aligned}$$

Therefore,

$$\sum_{m \in \mathbb{Z}} \Xi_0(2m) > \Xi_0(0) > 7 \log(2) - 1 > \frac{7\pi}{4} \log(2).$$

□

Appendix A. Number-theoretic estimates

The following lemmas are standard in number theory.

Lemma Appendix A.1. *For every $p \in \mathbb{N}$, we have*

$$\sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq m}} 1 = \frac{2^{-p} \pi^{\frac{p}{2}}}{\Gamma(1 + \frac{p}{2})} m^p + O(m^{p-1}), \quad \sum_{\substack{\mathbf{k} \in \mathbb{Z}^p \\ |\mathbf{k}| \leq m}} 1 = \frac{\pi^{\frac{p}{2}}}{\Gamma(1 + \frac{p}{2})} m^p + O(m^{p-1}).$$

Proof. We prove the first assertion by induction on p . Let $K = [0, 1]^p$ be the unit cube. For brevity, we denote p -tuple $(1, \dots, 1) = 1$. We have

$$\sum_{\substack{\mathbf{k} \in \mathbb{N}^p \\ |\mathbf{k}| \leq m}} 1 = \sum_{\substack{\mathbf{k} \in \mathbb{N}^p \\ |\mathbf{k}| \leq m}} \int_{K+\mathbf{k}-1} dt = \int_{C_m} dt, \quad \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq m}} 1 = \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq m}} \int_{K+\mathbf{k}} dt = \int_{B_m} dt,$$

where

$$C_m = \bigcup_{\substack{\mathbf{k} \in \mathbb{N}^p \\ |\mathbf{k}| \leq m}} (K + \mathbf{k} - 1) \subset \{t \in \mathbb{R}_+^p : |t| \leq m\} \subset \bigcup_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq m}} (K + \mathbf{k}) = B_m.$$

It follows immediately that

$$\sum_{\substack{\mathbf{k} \in \mathbb{N}^p \\ |\mathbf{k}| \leq m}} 1 \leq \int_{\substack{t \in \mathbb{R}_+^p \\ |t| \leq m}} dt \leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq m}} 1. \quad (\text{A.1})$$

It is clear that

$$\sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq m}} 1 - \sum_{\substack{\mathbf{k} \in \mathbb{N}^p \\ |\mathbf{k}| \leq m}} 1 \leq p \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^{p-1} \\ |\mathbf{k}| \leq m}} 1 = O(m^{p-1}), \quad (\text{A.2})$$

where we used induction with respect to p in the last equality. Combining (A.1) and (A.2), we infer that

$$\sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq m}} 1 = \int_{\substack{t \in \mathbb{R}_+^p \\ |t| \leq m}} dt + O(m^{p-1}) = \frac{2^{-p} \pi^{\frac{p}{2}}}{\Gamma(1 + \frac{p}{2})} m^p + O(m^{p-1}).$$

This concludes the proof of the first equality.

To see the second equality, note that

$$2^p \sum_{\substack{\mathbf{k} \in \mathbb{N}^p \\ |\mathbf{k}| \leq m}} 1 \leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}^p \\ |\mathbf{k}| \leq m}} 1 \leq 2^p \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq m}} 1.$$

The second equality follows now from (A.2). \square

Lemma Appendix A.2. *Let $\alpha_p : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+^p$ be a bijection. If the mapping $m \rightarrow |\alpha_p(m)|$ increases, then*

$$|\alpha_p(m)|^p = 2^p \pi^{-p/2} \Gamma(1 + \frac{p}{2}) m + O(m^{\frac{p-1}{p}}).$$

Let $\alpha_p : \mathbb{Z}_+ \rightarrow \mathbb{Z}^p$ be a bijection. If the mapping $m \rightarrow |\alpha_p(m)|$ increases, then

$$|\alpha_p(m)|^p = \pi^{-p/2} \Gamma(1 + \frac{p}{2}) m + O(m^{\frac{p-1}{p}}).$$

Proof. It follows from Lemma Appendix A.1 that

$$\begin{aligned} m &\leq \sum_{|\alpha_p(k)| \leq |\alpha_p(m)|} 1 = \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq |\alpha_p(m)|}} 1 \leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq \lfloor |\alpha_p(m)| \rfloor + 1}} 1 = \\ &= \frac{2^{-p} \pi^{\frac{p}{2}}}{\Gamma(1 + \frac{p}{2})} |\alpha_p(m)|^p + O(|\alpha_p(m)|^{p-1}) \end{aligned}$$

and

$$\begin{aligned} m &\geq \sum_{|\alpha_p(k)| < |\alpha_p(m)|} 1 = \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| < |\alpha_p(m)|}} 1 \leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^p \\ |\mathbf{k}| \leq \lfloor |\alpha_p(m)| \rfloor - 1}} 1 = \\ &= \frac{2^{-p} \pi^{\frac{p}{2}}}{\Gamma(1 + \frac{p}{2})} |\alpha_p(m)|^p + O(|\alpha_p(m)|^{p-1}). \end{aligned}$$

A combination of these estimates yields the first assertion and the proof of the second one is identical. \square

Lemma Appendix A.3. *For every $p \in \mathbb{N}$, we have $(1 - \Delta_p)^{-p/2} \in \mathcal{L}_{1,\infty}$. For every normalised trace φ on $\mathcal{L}_{1,\infty}$, we have*

$$\Gamma(1 + \frac{p}{2})\varphi((1 - \Delta_p)^{-p/2}) = \pi^{p/2}.$$

Proof. It follows from Lemma Appendix A.2 that

$$\mu(m, (1 - \Delta_p)^{-p/2}) = \frac{\pi^{p/2}}{\Gamma(1 + \frac{p}{2})} \frac{1}{m+1} + O((m+1)^{-1-\frac{1}{p}}).$$

Therefore,

$$\sum_{m=0}^n \mu(m, (1 - \Delta_p)^{-p/2}) = \frac{\pi^{p/2}}{\Gamma(1 + \frac{p}{2})} \log(n+1) + O(1).$$

The assertion follows from Theorem 2.3. \square

Appendix B. An easy counter-example to formula 1.1

In Theorem 1.12, we required that both operators $T_1(1 + D^2)^{-1/2}$ and $T_2(1 + D^2)^{-1/2}$ are universally measurable. In this appendix, we show that a simpler counter-example with $T_2 = 1$ does exist if the requirement of universal measurability of $T_1(1 + D_1)^{-p/2}$ is omitted. This gives a counter-example to the formula because one does not take into account the correction of the limiting process by powers (cf as in Lemma 1.9).

Lemma Appendix B.1. *There exist a $0 \leq T_1 \in \mathcal{L}(l_2)$, a universally measurable operator $(1 + D^2)^{-1/2} \in \mathcal{L}_{1,\infty}$ and a Dixmier trace⁸ Tr_ω such that*

$$\text{Tr}_\omega((T_1 \otimes 1)(1 + D^2 \otimes 1 + 1 \otimes D^2)^{-1}) \neq \frac{\pi}{4} \text{Tr}_\omega(T_1(1 + D^2)^{-1/2}).$$

Proof. Set $D = \text{diag}(\{k\}_{k \geq 0})$ and $T_1 = \text{diag}(\{x(k)\}_{k \geq 0})$ with $x = \chi_{\cup_m [n_{2m}, n_{2m+1})}$, where $\log(n_m) = o(\log(n_{m+1}))$ as $m \rightarrow \infty$. Suppose that for every Dixmier trace Tr_ω we have

$$\text{Tr}_\omega\left(\text{diag}\left(\left\{\frac{x(k)}{1 + k^2 + l^2}\right\}_{k,l \geq 0}\right)\right) = \frac{\pi}{4} \text{Tr}_\omega\left(\text{diag}\left(\left\{\frac{x(k)}{k+1}\right\}_{k \geq 0}\right)\right). \quad (\text{B.1})$$

In what follows, we omit diag to lighten the notations. Using definition (2.1) of Dixmier traces, we can equivalently rewrite (B.1) as

$$\lim_{n \rightarrow \omega} \frac{1}{\log(n+2)} \left(\sum_{i=0}^n \mu\left(i, \left\{\frac{x(k)}{1 + k^2 + l^2}\right\}_{k,l \geq 0}\right) - \frac{\pi}{4} \sum_{i=0}^n \mu\left(i, \left\{\frac{x(k)}{k+1}\right\}_{k \geq 0}\right) \right) = 0$$

⁸See (2.1) for the definition of Dixmier trace.

for every ultrafilter ω . Equivalently, we have

$$\sum_{i=0}^n \mu\left(i, \left\{\frac{x(k)}{1+k^2+l^2}\right\}_{k,l \geq 0}\right) - \frac{\pi}{4} \sum_{i=0}^n \mu\left(i, \left\{\frac{x(k)}{k+1}\right\}_{k \geq 0}\right) = o(\log(n)), \quad n \rightarrow \infty.$$

Lemma 5.2 states that

$$\sum_{i=0}^n \mu\left(i, \left\{\frac{x(k)}{1+k^2+l^2}\right\}_{k,l \geq 0}\right) - \sum_{k,l=0}^{\lfloor n^{1/2} \rfloor} \frac{x(k)}{1+k^2+l^2} = O(1),$$

while Lemma 5.1 states that

$$\sum_{i=0}^n \mu\left(i, \left\{\frac{x(k)}{k+1}\right\}_{k \geq 0}\right) - \sum_{k=0}^n \frac{x(k)}{k+1} = O(1).$$

We have

$$\begin{aligned} \sum_{k,l=0}^{\lfloor n^{1/2} \rfloor} \frac{x(k)}{1+k^2+l^2} &= \sum_{k=0}^{\lfloor n^{1/2} \rfloor} x(k) \sum_{l=0}^{\lfloor n^{1/2} \rfloor} \frac{1}{1+k^2+l^2} = \\ &= \sum_{k=0}^{\lfloor n^{1/2} \rfloor} x(k) \left(O\left(\frac{1}{1+k^2}\right) + \int_0^{n^{1/2}} \frac{dt}{1+t^2+k^2} \right) = \\ &= \sum_{k=0}^{\lfloor n^{1/2} \rfloor} \frac{x(k)}{(k^2+1)^{1/2}} \tan^{-1}\left(\left(\frac{n}{k^2+1}\right)^{\frac{1}{2}}\right) + O(1). \end{aligned}$$

Thus,

$$\frac{\pi}{4} \sum_{k=0}^n \frac{x(k)}{k+1} - \sum_{k=0}^{\lfloor n^{1/2} \rfloor} \frac{x(k)}{k+1} \tan^{-1}\left(\left(\frac{n}{k^2+1}\right)^{\frac{1}{2}}\right) = o(\log(n)), \quad n \rightarrow \infty. \quad (\text{B.2})$$

We now show that (B.2) actually fails. For $n = n_{2m}^2$, we have

$$\sum_{k=0}^n \frac{x(k)}{k+1} \geq \sum_{k=n_{2m}^2}^{n_{2m}^2} \frac{1}{k+1} = \log(n_{2m}) + O(1) = \frac{1}{2} \log(n) + O(1)$$

and, taking into account that x vanishes on the interval $[n_{2m-1}, n_{2m})$, we have

$$\sum_{k=0}^{\lfloor n^{1/2} \rfloor} \frac{x(k)}{k+1} \tan^{-1}\left(\left(\frac{n}{k^2+1}\right)^{\frac{1}{2}}\right) \leq \frac{\pi}{2} \sum_{k=0}^{n_{2m-1}} \frac{1}{k+1} = \frac{\pi}{2} \log(n_{2m-1}) + O(1) = o(\log(n)).$$

Hence, (B.2) fails for such x as $n = n_{2m}^2$ and $m \rightarrow \infty$. \square

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